Symmetry in Cartan language for geometric theories of gravity

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We present a recent definition of symmetry generating vector fields on manifolds equipped with a first-order reductive Cartan geometry. We apply this definition to a number of spacetime geometries used in gravity theories and show that it agrees with the usual notions of symmetry of affine, Riemann-Cartan, Riemannian, Weizenböck and Finsler spacetimes.

Keywords: Cartan geometry; symmetry; spacetime model

1. Definition

Let M be a manifold and $\varphi : \mathbb{R} \times M \to M$ a one-parameter group of diffeomorphisms generated by a vector field ξ on M. On the general linear frame bundle

$$\mathrm{GL}(M) = \bigcup_{x \in M} \{\text{linear bijections } f : \mathbb{R}^n \to T_xM\} \tag{1}$$

we define a one-parameter group of diffeomorphisms $\bar{\varphi} : \mathbb{R} \times \mathrm{GL}(M) \to \mathrm{GL}(M)$ by $\bar{\varphi}_t(f) = \varphi_{t*} \circ f$. This one-parameter group is generated by a vector field $\bar{\xi}$ on $\mathrm{GL}(M)$, which we call the frame bundle lift of ξ.

Let G be a Lie group with closed subgroup $H \subset G$, $\pi : P \to M$ a principal H-bundle with $P \subset \mathrm{GL}(M)$ and $A \in \Omega^1(P, \mathfrak{g})$ a Cartan connection which is first order reductive, i.e., the adjoint representation of H on the Lie algebra $\mathfrak{g}$ splits into subrepresentations $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ and the resulting representation on $\mathfrak{z}$ is faithful, and such that the $\mathfrak{z}$-valued part $e \in \Omega^1(P, \mathfrak{z})$ of A is the solder form on P. We call the Cartan geometry $(\pi : P \to M, A)$ *invariant*[1] under a vector field ξ on M if and only if the frame bundle lift $\bar{\xi}$ is tangent to P and its restriction to P preserves A, i.e., $\mathcal{L}_{\bar{\xi}} A = 0$.

2. Applications

We apply the notion of invariance defined above to a number of common spacetime geometries used in various models of gravity. These geometries can be written as first-order reductive Cartan geometries. In particular, we obtain the following notions of invariance under a vector field ξ:

- Affine geometry with connection Γ:
$$\mathcal{L}_{\xi}\Gamma = 0\,. \tag{2}$$
- Riemann-Cartan geometry with metric g and torsion T:
$$\mathcal{L}_{\xi}g = 0 \quad \wedge \quad \mathcal{L}_{\xi}T = 0\,. \tag{3}$$
- Riemannian geometry with metric g:
$$\mathcal{L}_{\xi}g = 0\,. \tag{4}$$
- Weizenböck geometry with tetrad e:
$$\mathcal{L}_{\xi}e = \lambda e\,, \tag{5}$$
where λ is a constant infinitesimal Lorentz transformation.
- Finsler geometry[2] with Finsler length function F:
$$\mathcal{L}_{\hat{\xi}}F = 0\,, \tag{6}$$
where $\hat{\xi}$ is the tangent bundle lift of ξ, i.e., the vector field on TM which is defined in analogy to the frame bundle lift $\bar{\xi}$, but replacing the diffeomorphisms $\bar{\varphi}_t$ with $\hat{\varphi}_t = \varphi_{t*}$.

These notions of invariance agree with the standard notions of invariance on the respective spacetime models.

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References

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